Counterexamples in Commutative Algebra

Mentors : Arkamouli Debnath , Dr. Jarod Alper

Student: Kaira Nair

Definitions

1. Commutative Ring : A ring R where a.b = b.a for all a,b $\in$ R
2. Zero Divisor : An element a $\neq$ 0 $\in$ R such that there exists b $\neq$ 0 with a.b=0
3. Integral Domain : A commutative ring with no zero divisors
4. Ideal : An ideal I in a commutative ring R is a subset such that if a,b $\in$ I then a+b $\in$ I and if a $\in$ I and r $\in$ R then a.r $\in$ I

Proving that ZMod6 is not an Integral Domain

We assume ZMod 6 is a domain and aim for a contradiction

```
theorem  not_domain : ¬(IsDomain (ZMod 6)) := by
  intro hDomain

  have h1 : (2: ZMod 6) ≠ (0: ZMod 6):= by decide
  have h2 : (3: ZMod 6) ≠ (0: ZMod 6) := by decide
  have h3 : (2*3 : ZMod 6) = (0 : ZMod 6) := by rfl
  -- Use eq_zero_or_eq_zero_of_mul_eq_zero directly
  rcases eq_zero_or_eq_zero_of_mul_eq_zero h3 with h_left | h_right
  -- Now we have our contradiction
  · exact h1 h_left   -- Contradicts 2 ≠ 0
  · exact h2 h_right  -- Contradicts 3 ≠ 0
```

h1: establishes that 2 ≠ 0 in ZMod 6
h2: establishes that 3 ≠ 0 in ZMod 6
h3: shows 2*3 = 0 in ZMod6

This tactic applies the property that in a domain, if a*b = 0, then a = 0 or b = 0

Proved by contradiction

(Trying to) Prove the quotient ring Q[x,y]/(xy) is not an Integral Domain

```
import Mathlib

abbrev PolyRing := MvPolynomial (Fin 2) ℚ

noncomputable def x : PolyRing := MvPolynomial.X 0
noncomputable def y : PolyRing := MvPolynomial.X 1

noncomputable def xyIdeal : Ideal PolyRing :=
  Ideal.span {x * y}
abbrev QuotRing := PolyRing / xyIdeal

noncomputable def x_bar : QuotRing := Ideal.Quotient.mk xyIdeal x
noncomputable def y_bar : QuotRing := Ideal.Quotient.mk xyIdeal y

#check QuotRing

theorem quotring_not_domain : ¬(IsDomain QuotRing) := by
  sorry
```

This defines a polynomial ring in 2 variables

These define the two variables as x and y.

These define the ideal generated by x.y and the quotient ring by modding out the ideal.

These define the quotient class of x and y in the quotient ring.

Unfortunately we were not able to write down a formal Lean proof yet.

Here are 2 reasons we couldn't exactly replicate the proof of the ZMod6 case.

1. Mathlib doesn't seem to already have the data that $\bar{x}$ and $\bar{y}$ are not equal to 0 in the ring $\mathbb{Q}[x,y]/(xy)$ unlike what it already knows about 2 and 3 not being equal to 0 in ZMod 6.
2. The fact that $\bar{x}.\bar{y} = 0$ in $\mathbb{Q}[x,y]/(xy)$ could not be proved by rfl tactic.